Implications of Research on Children's Understanding of Geometry

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In my mathematics methods course for preservice teachers, I ask my students to assess a group of elementary school students concerning their level of geometric reasoning. To do so, they use tasks that focus on assessing and extending students' geometric understandings. These open-ended tasks, along with a framework developed from research findings involving children's geometric reasoning, are described in this article. An important aspect of these tasks is that they focus on how students communicate their reasoning so that my preservice teachers can make more informed instructional decisions when planning a follow-up geometry lesson. Research on geometric reasoning has shown that a match between students' reasoning level and instructional tasks is crucial if meaningful learning is to occur (Crowley 1987).

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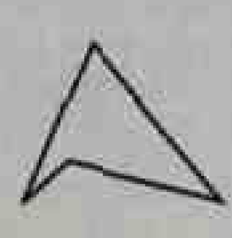
Recent Research on Geometric Reasoning

Much of the recent research on geometric reasoning has focused on the van Hiele model (Crowley 1987). As part of an ongoing research program in Cognitively Guided Instruction in Geometry, Lehrer and his colleagues (1994) have extended this knowledge base by refining and expanding the earlier levels of geometric reasoning commonly exhibited by primary school students, namely, visualization, analysis, and informal deduction. My preservice teachers examined the results from these tasks using this framework.

Geometric reasoning by resemblance

The first level at which children reason about shape is termed *resemblance.* Students at this level classify objects on the basis of their resemblance to other shapes and often rely on irrelevant characteristics of the object. Reasoning by resemblance can be classified as either *direct* or *indirect.* Using direct resemblance, students may classify a chevron



as a triangle, because it resembles a "pushed in" triangle. They may also rely on visual stereotypes, as commonly found in textbooks. For example, a "thin" rectangle may not be classified as a rectangle because it is "too thin" relative to the student's rectangular prototype of a door. With indirect resemblance, students may mentally modify a figure to represent another figure that is better known (Lehrer et al. 1994). For example, a chevron

could be considered a triangle by "pulling back" two of its sides until it looks like a triangle.

Geometric reasoning by attributes

As their reasoning matures, students begin to consider a shape's particular attributes or parts, perhaps by using informal language. However, students may not understand the relationships between these attributes. For example, students who do not understand the relationship between the angles and the number of sides of a figure would have to count its angles even though they know the number of its sides. Students who understand such relationships are reasoning at higher levels (Lehrer et al. 1994). At this level, students understand that the chevron is not a triangle but a quadrilateral because it has four sides.

Geometric reasoning by properties

At the next level of reasoning, students analyze a figure in terms of its properties and see relationships between these properties. For example, students at this level know that a square is also a parallelogram because it is a quadrilateral with both pairs of opposite sides being parallel. They also understand that changing a "critical" property of a given shape, such as the length of one side, changes the shape's classification and that changing a "noncritical attribute," such as its size, does not (Lehrer et al. 1994).

Assessment Tasks

My preservice teachers assess their students' levels of geometric reasoning using the following tasks. I asked the preservice teachers to focus especially on developing questions that encourage students to explain their reasoning. For several tasks, I have included examples of their students' thinking. This thinking has been analyzed using the framework described in this article, and follow-up tasks based on this thinking are proposed.

These tasks are open-ended and can be conducted in small groups or in a whole-group setting. Students should be asked to record their thinking on paper to further their own understanding and to help their teacher assess it.

The Polygon Comparison task

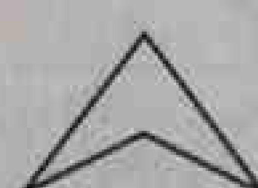
Task description. Students compare and contrast a trio of polygons (adapted from Lehrer et al. [1993]):

They are also asked to state or write which two figures are most alike and why. This task helps determine whether a student is focusing on a figure's appearance or its parts and properties. Lehrer et al. (1993) also suggest presenting this task using these trios:

and

Examples of student's thinking. When asked to compare these three objects,

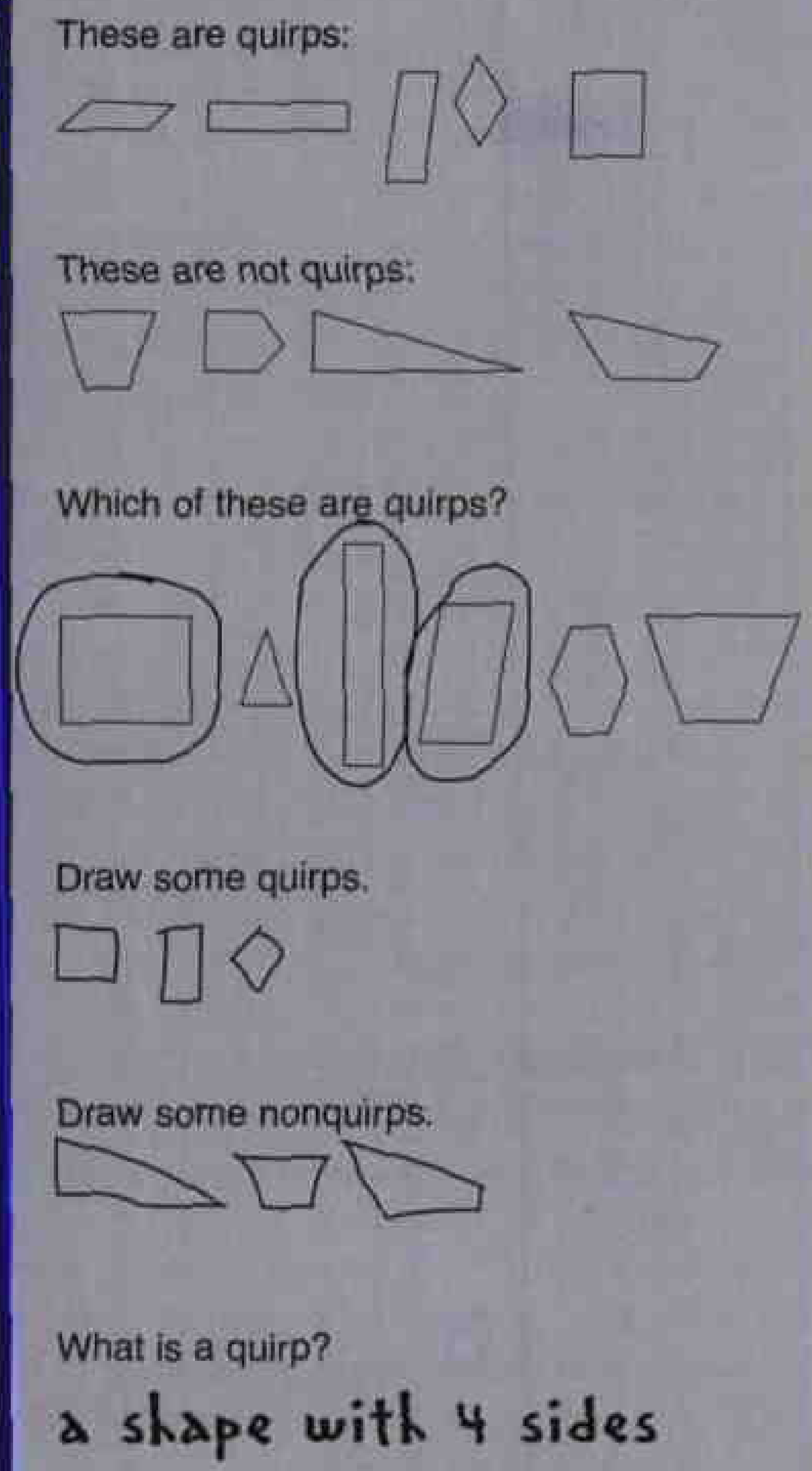
and to describe any similarities and differences among them, one second-grade student reasoned that the two figures that were most alike were the chevron and the triangle, explaining that "if you took the



and drew a line across it, it would become a ."

Such thinking is characteristic of indirect resemblance. Other students also believed that the second two figures in the trio were most alike. However, the reasoning involved direct resemblance to known geometric figures: "They are both triangles." For these students, appropriate follow-up tasks would include the Polygon Sort activity, which is described subsequently. This activity, especially when completed in a group setting, can

FIGURE 3 **Items examined as part of the Concept Card task**



cept cards for various types of polygons. Such a task can help a teacher determine to what extent students understand a particular type of polygon, as well as its relationships with other types of polygons.

Example of student's thinking. The student completing the concept card in **figure 3** was able to successfully identify and draw quirps, or parallelograms. However, he had more difficulty defining the shape fully. Since his difficulty centered on writing the definition, his teacher could follow up this task with the Polygon Definition task described previously. This extension activity could help the student focus more intently on the crucial attributes and properties of parallelograms.

Action Research Ideas

To consider the ideas discussed here, ask students to compare these three polygons:

The information gained from this particular trio can be helpful in determining whether students are using resemblance as the criterion for reasoning about polygons. On the basis of students' thinking elicited from this task, teachers can choose from among the other tasks described here. Several action research activities are suggested:

- After students have completed one or more tasks, teachers can consider and describe how students' backgrounds, characteristics, or interests may relate to their interpretations of the various tasks. Investigating how results from task assessments relate to individual student characteristics could help teachers choose possible remediation or extension activities.
- Teachers can examine how results from the described tasks relate to student performance on other mathematical assessment tasks. For example, descriptions of students' performance on geometric constructions using a geoboard could be compared with results on the Polygon Definition task to assess differences between visual and concrete abilities.
- Teachers may develop a miniature version of a case study for a student whose task assessment indicates that he or she is above or below a "norm." They could document change over time after completing a preestablished course of instruction.

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tions of various polygons' parts and properties. These tasks are described in the following sections. Students could also be asked to sort their polygons into various subclasses. For example, the students could be asked to further sort their quadrilaterals to discover some relationships among the figures. Another activity would include the Name and Property Cards task (Fuys, Geddes, and Tischler 1988). This task asks students to match name and property cards (see **fig. 2**) with the groups of polygons sorted previously. Because students may have sorted a particular polygon into subclasses, it is important to include several copies of each of the different name and property cards. For example, the "All angles are congruent" card would be appropriate for three different groups: squares, rectangles, and equilateral triangles.

The Polygon Definition task

Task description. This task helps identify students' conceptions or limited conceptions about polygons. Students are asked to define a given polygon and provide a rationale for their definition (adapted from Burger and Shaughnessy [1986]). To begin, each student is asked to write a definition for a given geometric figure, then the teacher leads a discussion of the definition. To help students' clarify their thinking, the teacher draws a figure to match the students' definition. If the teacher's drawing does not produce the intended result, then students can modify an inappropriate or incomplete definition. This task is most appropriate for students who have begun to reason using a figure's attributes. It requires students to focus on a figure's parts and properties while reaching group consensus on the minimum amount of information necessary to define a given geometric figure. Students can then complete the Name and Property Cards task described previously **(fig. 2).**

Example of student's thinking. When fifth-grade students were asked to define a rectangle, one student stated that "it has four sides. It's a polygon and all four sides would be right angles and has two sets of parallel lines and the only difference is two lines are shorter than two other lines." This definition describes the prototypical rectangle, since it includes properties not necessarily held by all rectangles. From this information, their teacher could follow up by confronting this misconception, perhaps by constructing a Concept Card using a wide variety of rectangles to help students discover the diversity of rectangles.

The Concept Card task

Task description. Students examine a card that displays examples and nonexamples of a geometric figure that has been given a fictitious name, say, *quirp* (adapted from Geddes and Fortunato [1993]). The students are then asked to identify quirps from a set of various figures and to give a definition of a quirp. An example of a concept card, along with one student's work, can be found in **figure 3.** This task is an "effective way to familiarize students with a concept and to make them aware of its distinguishing characteristics" (Geddes and Fortunato 1993, 204). It also requires students to move beyond reliance on rotely acquired facts about standard geometric figures. Teachers can construct different concept cards for various types of polygons and their subclasses, giving each type of polygon a different, fictitious name. A concept card can be made by first choosing one type of polygon and giving it a new, fictitious name; drawing examples of that polygon and any polygons in its subclasses; and then constructing nonexamples to include classes of polygons that distinguish other types from the types shown on the concept card. For example, if constructing a concept card for rectangles, examples should include both rectangles and squares; the nonexamples should include trapezoids, parallelograms, rhombuses, and kites, as well as quadrilaterals that do not fit into any of these classes, such as a chevron. An additional example of a concept card can be found in Geddes and Fortunato (1993, 208). As a follow-up activity, students can also be asked to construct their own con-

FIGURE 2

Information in the Property and Name Cards task (Fuys, Geddes, and Tischler 1988)

- Triangle
- Rectangle
- Rhombus
- Octagon
- Equilateral Triangle
- Kite
- Trapezoid
- All of its angles are congruent
- Has one pair of parallel sides
- Closed
- All of its sides are congruent
- Concave
- Has two pairs of parallel sides
- Acute Triangle
- Has opposite sides congruent
- Has no congruent sides
- Has four congruent sides but does not have to

encourage students to focus on a figure's attributes, thus attaining the next higher level of reasoning.

When examining a different trio of polygons,

a pair of fourth-grade students studied the polygons' attributes. They identified the first two figures as having four sides, which indicated that these students were using attributes of the polygons as the primary distinguishing characteristic. Several other students reasoned in a similar manner. For these students, tasks that helped them focus more explicitly on polygons' parts and properties would be especially appropriate. One such task would be the Name and Property Cards task to be described subsequently.

The Polygon Sort task

Task description. Students are asked to sort a set of cardboard or paper polygonal cutouts in a meaningful way and give a rationale for their sort (Fuys, Geddes, and Tischler 1998). If students are unsure of how to proceed, they should be asked how the various polygonal cutouts are different and to sort the figures accordingly. This task can help reveal whether students' reasoning involves appearance, attributes, or properties.

To best determine the level of students' reasoning about various subclasses of a particular polygon, examples of all the various subclasses should be included in the set of polygonal cutouts. For example, several cutout examples having the shape of equilateral, isosceles, and scalene triangles should be included. Also important for inclusion are examples of "typical" polygons, as usually found in textbooks, and "atypical" polygons, such as a "thin" triangle (see **fig. 1**). The greater the variety of polygon cutouts, the greater the potential to learn about students' reasoning. To help the teacher assess the students' reasoning, the students can be asked to record on paper their sorting rationale, as well as to draw and name the different figure categories.

Example of students' thinking. When sorting figures like those shown in **figure 1,** students used various rationales. Many students used the number of sides as their criterion, which indicates reasoning based on attributes. For these students, assigning both the Polygon Definition and Concept Card tasks would be appropriate because these tasks encourage students to begin to formulate defini-

FIGURE 1

Figures used for the Polygon Sort task